3. (T → T)-branch (branch 4)

This vibration is exactly transverse for both principal directions and therefore characterized by only one quasi-normal coordinate ($\bar{Q}_{\perp 4}$ respectively $\bar{Q}_{\parallel 1}$). Going from $\vartheta = 0^\circ$ to $\vartheta = 90^\circ$ these two coordinates interchange their predominance. In the intermediate range $0^\circ < \vartheta < 90^\circ$ there is a weak coupling of all other coordinates to the predominant coordinates $\bar{Q}_{\perp 4}$ and $\bar{Q}_{\parallel 1}$. The polarisation $\boldsymbol{P}$ is nearly transverse over the whole range of ϑ, the electric field is very weak compared to the other branches because it is exactly longitudinal.

4. (L → L)-branch (branch 13)

Because this vibration is exactly longitudinal for both principal directions and the electric field therefore being relatively strong over the whole range of ϑ, there is a strong coupling of the quasi-normal coordinates for all directions. Only for $\vartheta = 0^\circ$ and $\vartheta = 90^\circ$ all $\bar{Q}_{\parallel k}$ and $\bar{Q}_{\perp j}$, respectively, vanish for reasons of symmetry. The polarisation $\boldsymbol{P}$ is nearly longitudinal for all angles ϑ.

All numerical calculations were carried out on the IBM computer 360/50 of the Universität Münster.

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A Simple Algebraic Calculation of the Resolvent in the V Θ-Sector of the Lee Model

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It is shown that the resolvent in the $V\Theta$-sector of the Lee model may be expressed by simple algebraic operations in terms of known matrix elements between states of the V-sector.

1. Introduction

It is known that the stationary states in the $V\Theta$-sector of the LEE model [1] may be obtained from the solution of the KÄLLÉN-PAULI integral equation [2], which has been determined in closed form by several authors [3–8] using the theory of analytic functions. On the other hand, FIVEL [9] has shown that it is also possible to solve the eigenvalue problem in the $V\Theta$-sector in a completely algebraic way. His method consists in applying a unitary transformation to the HAMILTONIAN such that a separable potential problem arises which may be reduced to a system of linear equations. The calculations required are, however, rather complicated, especially as regards the comparison of the result with that of the aforementioned authors.

In the present paper we wish to show that by extremely simple algebraic operations the resolvent in the $V\Theta$-sector may be expressed in terms of known matrix elements from the V-sector. In fact, the problem reduces to solving a linear integral equation with a degenerate kernel which is equivalent to one linear equation for one unknown constant. In comparing our result with the standard form of the solution we have to use some relations between analytic functions, which are however rather trivial.

We need not distinguish explicitly between the cases of a stable or an unstable V-particle. In order

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to make our considerations as transparent as possible, we avoid renormalization problems by introducing a cut-off in momentum space. As our method is especially adapted to the calculation of the resolvent, we shall not discuss stationary states and the scattering matrix. A purely algebraic calculation of these quantities may be found in the independent work of BOLSTERLI[10] which came to our attention after finishing the manuscript. Bolsterli's procedure resembles ours in that multiplication by operators a_k^*, $V^* N$ is used for passing from the V-sector to the $V\Theta$-sector.

2. Calculation of the Resolvent by Conventional Methods

2.1. The Hamiltonian

The resolvent is defined by $(x-H)^{-1}$, x being a complex variable which may assume arbitrary values outside the spectrum of the Hamiltonian H. It can be determined from the equation

$$1 = (x-H)^{-1}(x-H) \tag{1}$$

by taking matrix elements between unperturbed states.

We consider the usual Hamiltonian

$$H = H_0 + H_1\,,\; H_0 := M_0 V^* V + m N^* N + \int \mathrm{d}k\, k_0\, a^k a_k\,,\; H_1 := V^* N A + N^* V A^*\,, \tag{2}$$

$$A := \int \mathrm{d}k\, v^k a_k\,,\; A^* := \int \mathrm{d}k\, v_k a^k,\; v_k := (v^k)^*,\; a^k := (a_k)^*. \tag{3}$$

M_0 is the unperturbed mass of the V-particle, m the mass of the N particle, k_0 is the relativistic mass (energy)

$$k_0 := {}_+\sqrt{\mu^2 + k^2} \tag{4}$$

of the Θ particle, μ being the rest mass, k the 3-momentum of the latter. Volume elements in 3-dimensional momentum space will be denoted by $\mathrm{d}k$, $\mathrm{d}k'$, $\mathrm{d}l$ etc., the relativistic energies derived from 3-momenta k', l etc. by formulas analogous to (4) will be written k_0', l_0, etc. The asterisk * has the usual meaning of complex conjugate number or hermitean adjoint operator; in quantities concerning the Θ particle it may be avoided by raising or lowering indices [see, e. g. the quantities v_k, a^k defined in (3)]. The corresponding tensor calculus will not be used here, apart from writing for the Dirac δ-function in momentum space

$$\delta_l^k := \delta(k-l) = \delta_k^l\,; \tag{5}$$

in particular, an "integration convention" will not be understood [cf. the operators A, A^*, Eq. (3)]. We here regard the position of indices merely as a means for simplifying notation and checking formulas.

The function v^k of 3-momentum k entering the Hamiltonian according to (2), (3) is to a large extent arbitrary; we assume that it is so smooth and decreases so rapidly for large $|k|$ that all integrals and operators exist which occur in the following.

[10] M. BOLSTERLI, Phys. Rev. **166**, 1760 [1968].

As usual, V, N, a_k are destruction operators for the V, N, Θ particles, respectively. The Fock space vacuum ω satisfies

$$V\omega = N\omega = a_k\,\omega = 0, \quad (\omega,\omega) = 1. \tag{6}$$

The commutation relations are

$$\{V, V^*\} = \{N, N^*\} = 1, \quad [a_k, a^l] = \delta_k^l\,, \tag{7}$$

al other commutators (anticommutators) vanishing. It is well known that the operators of "baryon number"

$$B := V^* V + N^* N \tag{8}$$

and "charge"

$$Q := V^* V + \int \mathrm{d}k\, a^k a_k \tag{9}$$

commute among themselves and with the Hamiltonian H.

2.2. The Resolvent in the V-sector

The states $\{V^*\omega, a^k N^*\omega\}$ are an orthonormal basis of the V-sector $\mathfrak{H}_1$ $(B=1, Q=1)$ and, furthermore, eigenstates of H_0. With respect to this basis, there are four types of matrix elements of the resolvent:

$$\begin{aligned} q(x) &:= (V^*\omega, (x-H)^{-1} V^*\omega)\,,\\ q_l(x) &:= (a^l N^*\omega, (x-H)^{-1} V^*\omega)\,,\\ q^k(x) &= (V^*\omega, (x-H)^{-1} a^k N^*\omega)\,,\\ q_l^k(x) &= (a^l N^*\omega, (x-H)^{-1} a^k N^*\omega)\,. \end{aligned} \tag{10}$$

Using

$$\begin{aligned} (V^*\omega, H V^*\omega) &= M_0\,,\\ (a^l N^*\omega, H V^*\omega) &= v_l\,,\\ (V^*\omega, H a^k \mathrm{N}^*\omega) &= v^k,\\ (a^l N^*\omega, H a^k N^*\omega) &= (m+k_0)\,\delta_l^k, \end{aligned} \tag{11}$$

Equation (1) in the V-sector is easily seen to be equivalent to the folowing set of equations for the quantities (10):

$$1 = (x-M_0)\, q(x) - \int \mathrm{d}k\, v_k\, q^k(x), \quad (12\,a)$$

$$0 = -v^k\, q(x) + (x-m-k_0)\, q^k(x), \quad (12\,b)$$

$$0 = (x-M_0)\, q_l(x) - \int \mathrm{d}k\, v_k\, q_l^k(x), \quad (12\,c)$$

$$\delta_l^k = -v^k\, q_l(x) + (x-m-k_0)\, q_l^k(x). \quad (12\,d)$$

The solution is obtained by trivial algebraic operations (similar to those in solving the eigenvalue problem of H in $\mathfrak{H}_1$), the result being

$$\begin{aligned} &q(x) = h(x-m)^{-1},\ q^k(x) = v^k(x-m-k_0)^{-1} h(x-m)^{-1},\\ &q_l(x) = v_l(x-m-l_0)^{-1}\, h(x-m)^{-1},\\ &q_l^k(x) = \delta_l^k\,(x-m-k_0)^{-1} + v^k v_l(x-m-k_0)^{-1}(x-m-l_0)^{-1} h(x-m)^{-1}, \end{aligned} \quad (13)$$

where

$$h(x) := x+m-M_0+\int \mathrm{d}k\, v_k\, v^k(k_0-x)^{-1} \quad (14)$$

is an auxiliary function well known from the solution of the eigenvalue problem [11, 12]. It should be noted that by (10) and (13)

$$(V^*\omega, (x-H)^{-1} V^*\omega) = h(x-m)^{-1}, \quad (15)$$

with *unrenormalized* operators V^* in the matrix element on the left, so that the function (14) differs from that denoted by the same symbol in Ref. [12] (p. 367) (provided the V particle is stable) by a multiplicative renormalization constant.

2.3. The Resolvent in the $V\,\Theta$-sector

The states $\{a^l\, V^*\omega,\ a^{k_1} a^{k_2} N^*\omega\}$ are an orthonormal basis of the $V\,\Theta$-sector $\mathfrak{H}_2$ $(B=1,\ Q=2)$, and, furthermore, eigenstates of H_0. There are again four types of matrix elements of the resolvent:

$$\begin{aligned} &r_l^k(x) := (a^l V^*\omega, (x-H)^{-1}\, a^k V^*\omega),\quad r_l^{k_1k_2}(x) := (a^l V^*\omega, (x-H)^{-1}\, a^{k_1} a^{k_2} N^*\omega),\\ &r_{l_1l_2}^k(x) := (a^{l_1} a^{l_2} N^*\omega,\ (x-H)^{-1} a^k V^*\omega),\quad r_{l_1l_2}^{k_1k_2}(x) := (a^{l_1} a^{l_2} N^*\omega,\ (x-H)^{-1} a^{k_1} a^{k_2} N^*\omega) \end{aligned} \quad (16)$$

with the symmetry properties

$$r_l^{k_1k_2}(x) = r_l^{k_2k_1}(x),\quad r_{l_1l_2}^k(x) = r_{l_2l_1}^k(x),\quad r_{l_1l_2}^{k_1k_2}(x) = r_{l_1l_2}^{k_2k_1}(x) = r_{l_2l_1}^{k_1k_2}(x) = r_{l_2l_1}^{k_2k_1}(x), \quad (17)$$

$$r_l^k(x)^* = r_k^l(x^*),\quad r_l^{k_1k_2}(x)^* = r_{k_1k_2}^l(x^*),\quad r_{l_1l_2}^{k_1k_2}(x)^* = r_{k_1k_2}^{l_1l_2}(x^*). \quad (18)$$

Using the matrix elements of H in the $V\,\Theta$-sector

$$\begin{aligned} &(a^l V^*\omega, H\, a^k V^*\omega) = (M_0+k_0)\,\delta_l^k,\quad (a^l V^*\omega, H\, a^{k_1} a^{k_2} N^*\omega) = v^{k_1}\delta_l^{k_2} + v^{k_2}\delta_l^{k_1},\\ &(a^{l_1} a^{l_2} N^*\omega, H\, a^k V^*\omega) = v_{l_1}\,\delta_{l_2}^k + v_{l_2}\,\delta_{l_1}^k,\\ &(a^{l_1} a^{l_2} N^*\omega, H\, a^{k_1} a^{k_2} N^*\omega) = (m+k_{10}+k_{20})\,(\delta_{l_1}^{k_1}\delta_{l_2}^{k_2} + \delta_{l_2}^{k_1}\delta_{l_1}^{k_2}), \end{aligned} \quad (19)$$

Equation (1) in the $V\,\Theta$-sector is found to be equivalent to the following set of equations for the quantities (16):

$$\delta_l^k = (x-M_0-k_0)\, r_l^k(x) - \int \mathrm{d}k'\, v_{k'}\, r_l^{kk'}(x), \quad (20\,a)$$

$$0 = -v^{k_1} r_l^{k_2}(x) - v^{k_2} r_l^{k_1}(x) + (x-m-k_{10}-k_{20})\, r_l^{k_1k_2}(x), \quad (20\,b)$$

$$0 = (x-M_0-k_0)\, r_{l_1l_2}^k(x) - \int \mathrm{d}k'\, v_{k'}\, r_{l_1l_2}^{kk'}(x), \quad (20\,c)$$

$$\delta_{l_1}^{k_1}\delta_{l_2}^{k_2} + \delta_{l_2}^{k_1}\delta_{l_1}^{k_2} = -v^{k_1} r_{l_1l_2}^{k_2}(x) - v^{k_2} r_{l_1l_2}^{k_1}(x) + (x-m-k_{10}-k_{20})\, r_{l_1l_2}^{k_1k_2}(x). \quad (20\,d)$$

These equations can be solved by first expressing the resolvent matrix elements (16) with 3 or 4 indices in terms of $r_l^k(x)$, so that only the latter remains to be determined.

From (20 b) we have

$$r_l^{k_1k_2}(x) = (x-m-k_{10}-k_{20})^{-1}\,(v^{k_1} r_l^{k_2}(x) + v^{k_2} r_l^{k_1}(x)). \quad (21)$$

Using (18), it follows that

$$r_{l_1l_2}^k(x) = r_k^{l_1l_2}(x^*)^* = (x-m-l_{10}-l_{20})^{-1}\,(v_{l_1} r_{l_2}^k(x) + v_{l_2} r_{l_1}^k(x)). \quad (22)$$

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[12] S. S. SCHWEBER, An Introduction to Relativistic Quantum Field Theory, Row, Peterson & Co., New York 1961.

Inserting into (20 d), we obtain

$$r_{l_1 l_2}^{k_1 k_2}(x) = (x-m-k_{10}-k_{20})^{-1}(\delta_{l_1}^{k_1}\delta_{l_2}^{k_2}+\delta_{l_2}^{k_1}\delta_{l_1}^{k_2}) + (x-m-k_{10}-k_{20})^{-1}(x-m-l_{10}-l_{20})^{-1} \cdot \left(v^{k_1} v_{l_1} r_{l_2}^{k_2}(x) + v^{k_1} v_{l_2} r_{l_1}^{k_2}(x) + v^{k_2} v_{l_1} r_{l_2}^{k_1}(x) + v^{k_2} v_{l_2} r_{l_1}^{k_1}(x)\right). \tag{23}$$

The remaining unknown matrix element $r_l^k(x)$ can be calculated in a way which resembles the determination of the eigenstates of the Hamiltonian in $\mathfrak{H}_2$, i. e. by solving an integral equation of the KÄLLÉN-PAULI type [2–8]. This "conventional" method requires the usual sophisticated considerations on analytic functions of an energy parameter in the complex domain. We shall not give any details here [13] but merely derive the integral equation.

Insertion of (21) into (20 a) yields, using the definition (14) of $h(x)$:

$$\delta_l^k = h(x-m-k_0)\, r_l^k(x) - v^k \int dk'\, v_{k'}(x-m-k_0-k_0')^{-1} r_l^{k'}(x), \tag{24}$$

which is an integral equation for $r_l^k(x)$ as a function of the *superscript*. Replacing x by x^* and taking the complex conjugate, we obtain [cf. (18)]

$$\delta_k^l = h(x-m-k_0)\, r_k^l(x) - v_k \int dk'\, v^{k'}(x-m-k_0-k_0')^{-1} r_{k'}^l(x), \tag{25}$$

which is an integral equation with respect to the *subscript*. The δ-function can be removed by considering the following auxiliary function of a complex variable y

$$\varphi^l(y,x) := \int dk'\, v^{k'}(y-k_0')^{-1} r_{k'}^l(x+m), \tag{26}$$

which determines r_k^l according to (25):

$$r_k^l(x) = h(x-m-k_0)^{-1}\{\delta_k^l + v_k\, \varphi^l(x-m-k_0, x-m)\}. \tag{27}$$

Inserting this into (26), we get an integral equation of the Källén-Pauli type for φ as a function of its first argument:

$$\varphi^l(y,x) = v^l(y-l_0)^{-1} h(x-l_0)^{-1} + \int dk'\, |v_k'|^2 (y-k_0')^{-1} h(x-k_0')^{-1} \varphi^l(x-k_0', x). \tag{28}$$

Under reasonable assumptions the solution is unique and reads [13]

$$\varphi^l(y,x) = v^l h(y)\left\{-\frac{2}{D(x)} N(x-l_0,x)\, N(y,x) + (y-l_0)^{-1}[N(x-l_0,x)+N(y,x)] + (y+l_0-x)^{-1}[N(x-l_0,x)-N(y,x)]\right\}, \tag{29}$$

where

$$N(y,x) := \int_{-\infty}^{+\infty} dy'\,(y-y')^{-1} h(x-y')^{-1} \varrho_1(y'), \tag{30}$$

$$D(x) := \lim_{y\to\infty} y\, N(y,x) = \int_{-\infty}^{+\infty} dy'\, h(x-y')^{-1} \varrho_1(y'), \tag{31}$$

$\varrho_1(x)$ being the (conveniently normalized) discontinuity of $h(x)^{-1}$ across the real axis (see Appendix 1). By inserting (29) into (27) we finally obtain (interchanging k and l):

$$r_l^k(x+m) = \delta_l^k h(x-k_0)^{-1} + v^k v_l \left\{-\frac{2}{D(x)} N(x-k_0,x)\, N(x-l_0,x) + (k_0-l_0)^{-1}[N(x-k_0,x)-N(x-l_0,x)] + (x-k_0-l_0)^{-1}[N(x-k_0,x)+N(x-l_0,x)]\right\}. \tag{32}$$

3. Algebraic Calculation of the Resolvent in the V Θ-Sector

3.1. Modified Resolvents Raising the Charge

After these preliminaries, we wish to show that the resolvent in the $V\Theta$-sector can be obtained by purely algebraic operations. It is only in comparing the result with Eq. (32) that relations between analytic functions have to be used.

[13] K. HELMERS, unpublished work.

We start by considering the "modified resolvents"

$$Q^v(x) := (x-H)^{-1} V^* N, \quad Q^k(x) := (x-H)^{-1} a^k. \tag{33}$$

These operators clearly map the $V\,\Theta^z$-sector $\mathfrak{H}_{z+1}$ $(B=1,\ Q=z+1)$ into the $V\,\Theta^{z+1}$-sector $\mathfrak{H}_{z+2}$ $(B=1,\ Q=z+2)$ (i. e. they increase the charge by one unit while the baryon number B is left unchanged). Once this mapping is known explicitly, the resolvent is known in the sector $\mathfrak{H}_{z+2}$. We are mainly interested in the case $z=0$ (transition from the V-sector to the $V\,\Theta$-sector); in this subsection, however, we leave z unspecified.

In order to find algebraic relationships between the operators (33), we prove a simple lemma (for a special case, see SCHWEBER [12], p. 359).

Lemma: Let $H=H_0+H_1$ be an operator, x and λ complex numbers such that $(x-H)^{-1}$ and $(x-\lambda-H)^{-1}$ exist, and C an operator satisfying

$$[H_0, C] = \lambda\, C, \tag{34}$$

then

$$(x-H)^{-1} C = C(x-\lambda-H)^{-1} + (x-H)^{-1}[H_1, C]\,(x-\lambda-H)^{-1}. \tag{35}$$

Proof:

$$(H-x)\,C - C(H-x) = [H-x, C] = [H, C] = \lambda\, C + [H_1, C]\,, \tag{36}$$

thus

$$C(x-H-\lambda) = (x-H)\,C + [H_1, C]\,, \tag{37}$$

from which (35) results after multiplication by $(x-H)^{-1}$ from the left, and by $(x-\lambda-H)^{-1}$ from the right. QED.

We now apply the lemma to the Lee model, the operators H, H_0, H_1 being defined by (2).

1) For $C=a^k$, we have $\lambda=k_0$, $[H_1, C]=[H_1, a^k]=V^* N v^k$, and (35) reads

$$(x-H)^{-1} a^k = a^k(x-k_0-H)^{-1} + v^k(x-H)^{-1} V^* N\,(x-k_0-H)^{-1}. \tag{38}$$

2) For $C=V^* N$, we have $\lambda=M_0-m$, $[H_1, C]=[H_1, V^* N]=(N^* N-V^* V)\,A^*$, and (35) reads

$$(x-H)^{-1} V^* N = V^* N\,(x-M_0+m-H)^{-1} + (x-H)^{-1}\,(N^* N-V^* V)\,A^*(x-M_0+m-H)^{-1}. \tag{39}$$

By using the definition of the operators A^*, Q^v, Q^k, we may rewrite (38) and (39) as follows:

$$Q^k(x) = \{a^k + v^k\,Q^v(x)\}\,(x-k_0-H)^{-1}, \tag{40 a}$$

$$Q^v(x) = \{V^* N + \int \mathrm{d}k\, v_k\, Q^k(x)\,(N^* N-V^* V)\}\,(x-M_0+m-H)^{-1}. \tag{40 b}$$

Elimination of $Q^k(x)$ yields an "algebraic" equation for $Q_v(x)$:

$$Q^v(x)\{(x-M_0+m-H) - \int \mathrm{d}k\, v_k\, v^k(x-k_0-H)^{-1}\,(N^* N-V^* V)\}$$
$$= V^* N + \int \mathrm{d}k\, v_k\, a^k\,(x-k_0-H)^{-1}\,(N^* N-V^* V)\,. \tag{41}$$

It will be noted that the operator multiplying $Q^v(x)$ from the right resembles (but does not equal) the operator $h(x-H)$ obtained from (14) by substituting the operator $x-H$ for the number x. It is this observation which stimulated the present work.

It is convenient to multiply (41) from the right by $N^* N-V^* V=B-2\,V^* V$ and to use the trivial identities

$$V^* N(N^* N-V^* V) = V^* N\,, \qquad (N^* N-V^* V)^2 = B^2 + 4\,V^* V(1-B)\,, \tag{42}$$

the latter operator being equivalent to the unit operator for $B=1$. Then

$$Q^v(x)\,J(x-H) = V^* N + \int \mathrm{d}k\, v_k\, a^k(x-k_0-H)^{-1}\,(B^2+4\,V^* V(1-B))\,, \tag{43}$$

where

$$J(x-H) := (x-H+m-M_0)\,(B-2\,V^* V) + \int \mathrm{d}k\, v_k\, v^k(k_0-x+H)^{-1}\,(B^2+4\,V^* V(1-B)). \tag{44}$$

Suppose the resolvent is known in the sector $\mathfrak{H}_{z+1}$. Then the right-hand side of Eq. (43) is know in this sector, and so is the operator $J(x-H)$. Furthermore, this operator leaves the space $\mathfrak{H}_{z+1}$ invariant, and the same is true for its inverse $J(x-H)^{-1}$ (if the latter exists, which will be verified below by explicit calculation). Thus if we succeed to determine $J(x-H)^{-1}$ in $\mathfrak{H}_{z+1}$, we know $Q^v(x)$ in this sector, hence $Q^k(x)$ by (40 a), and by the argument given above, the resolvent is known in the next higher sector $\mathfrak{H}_{z+2}$.

3.2. *Inversion of the Operator $J(x-H)$ in the V-sector*

Let $u \in \mathfrak{H}_{z+1}$ be a given vector. We wish to calculate the vector

$$\psi := J(x-H)^{-1} u \tag{45}$$

(which is also in $\mathfrak{H}_{z+1}$) from

$$J(x-H)\,\psi = u\,. \tag{46}$$

Since $B\psi = \psi$, this equation can be written as follows [see (44)]:

$$\begin{aligned} u &= (x-H+m-M_0)(1-2V^*V)\,\psi + \int dk\, v_k\, v^k (k_0 - x + H)^{-1}\psi \\ &= h(x-H)\,\psi - 2(x-H+m-M_0)\,V^*V\psi\,. \end{aligned} \tag{47}$$

In the V-sector ($u \in \mathfrak{H}_1$, $\psi \in \mathfrak{H}_1$) this equation becomes very simple. In the following, we restrict ourselves to this case. Then

$$V^*V\psi = \lambda V^*\omega\,, \qquad \lambda := (V^*\omega, \psi)\,. \tag{48}$$

By (47), the unknown vector ψ can be expressed by the given vector u and the unknown number λ:

$$\psi = h(x-H)^{-1}\{u + 2\lambda(x-H+m-M_0)\,V^*\omega\}\,. \tag{49}$$

Forming the scalar product with $V^*\omega$, we obtain a linear equation for λ:

$$\lambda = (V^*\omega, h(x-H)^{-1}u) + 2\lambda\,(V^*\omega, h(x-H)^{-1}(x-H+m-M_0)\,V^*\omega)\,, \tag{50}$$

the solution being

$$\lambda = (V^*\omega, h(x-H)^{-1}u)\,d(x-m)^{-1}\,, \tag{51}$$

where

$$d(x-m) := 1 - 2\,(V^*\omega, h(x-H)^{-1}(x-H+m-M_0)\,V^*\omega)\,. \tag{52}$$

Substitution of (51) into (49) yields the solution of (46):

$$\psi = J(x-H)^{-1}u = h(x-H)^{-1}\{u + 2\,d(x-m)^{-1}(V^*\omega, h(x-H)^{-1}u)\,(x-H+m-M_0)\,V^*\omega\}\,, \tag{53}$$

hence the inverse of $J(x-H)$ is known in the V-sector. Equation (43) then allows to calculate (for arbitrary $u \in \mathfrak{H}_1$):

$$Q^v(x)\,u = \{V^*N + \int dk\, v_k\, a^k (x-H-k_0)^{-1}\}\,J(x-H)^{-1}u\,. \tag{54}$$

The last two formulas suffice to calculate the resolvent in the $V\Theta$-sector. Due to the algebraic relations (21) – (23), only the simplest matrix element $r_l^k(x)$ need be determined. By definition [see (16)],

$$r_l^k(x) = (a^l V^*\omega, (x-H)^{-1} a^k V^*\omega) = (a^l V^*\omega, Q^v(x)\, a^k N^*\omega)\,, \tag{55}$$

where the definition (33) of $Q^v(x)$ and the identity $NN^*\omega = \omega$ have been used. In (54), we put

$$u = a^k N^*\omega\,, \tag{56}$$

and obtain

$$\begin{aligned} r_l^k(x) &= (a^l V^*\omega, V^*N\,J(x-H)^{-1}u) + \int dk'\, v_{k'}\,(a^l V^*\omega, a^{k'}(x-H-k_0')^{-1}J(x-H)^{-1}u) \\ &= (a^l N^*\omega, J(x-H)^{-1}u) + v_l\,(V^*\omega, (x-H-l_0)^{-1}J(x-H)^{-1}u)\,. \end{aligned} \tag{57}$$

By (53), the desired expression for $r_l^k(x)$ is a sum of four terms:

$$r_l^k(x) = a + b + c + d\,, \tag{58}$$

where

$$\begin{aligned} a &= (a^l N^*\omega, h(x-H)^{-1}u)\,, \\ b &= 2\,d(x-m)^{-1}(V^*\omega, h(x-H)^{-1}u)\,(a^l N^*\omega, h(x-H)^{-1}(x-H+m-M_0)\,V^*\omega)\,, \\ c &= v_l\,(V^*\omega, (x-H-l_0)^{-1}(h(x-H)^{-1}u))\,, \\ d &= v_l\,2\,d(x-m)^{-1}(V^*\omega, h(x-H)^{-1}u)\,(V^*\omega, (x-H-l_0)^{-1}h(x-H)^{-1}(x-H+m-M_0)\,V^*\omega) \end{aligned} \tag{59}$$

with u given by (56). Consequently, $r_l^k(x)$ is known if the matrix elements of certain functions of the Hamiltonian such as $h(x-H)^{-1}$, $h(x-H)^{-1}(x-H+m-M_0)$ etc. between the states $V^*\omega$, $a^{k'}N^*\omega$ of the V-sector are known. These matrix elements can be evaluated by first expressing the operator functions of $(x-H)$ as "sums" of terms $(\xi-H)^{-1}$ (ξ being a number), which amounts to finding the "partial fraction decomposition" (i. e. the representation by Cauchy integrals) of certain analytic functions such as $h(x)^{-1}$, $h(x)^{-1}(x+m-M_0)$ etc. Next, matrix elements of $(\xi-H)^{-1}$ have to be calculated, which are known from Eqs. (13). In this way, $r_l^k(x)$ is explicitly expressed in terms of some integrals which resemble the integrals N and D [see Eqs. (30), (31)] but are not identical with them. Finally, the form (32) for $r_l^k(x)$ is obtained by using some simple relations between the various integrals. In view of the fact that the procedure is somewhat lengthy (for a detailed account, see Ref. [14]), we outline it in Appendix 2.

Appendix

1. *A Representation for the Function* $h(x)^{-1}$

We wish to explain the quantity $\varrho_1(x)$ occurring in the integrals N [Eq. (30)] and D [Eq. (31)] as well as in the decomposition into partial fractions (PFD) of the analytic function $h(x)^{-1}$.

The function $h(x)$ defined in (14) is known to have the following properties:

(a) $h(x)$ is regular in x apart from a cut $\mu < x < \infty$ along the real x axis.

(b) For large x in the cut x-plane, $h(x) \approx x$. This is true under our assumption that the function v^k (and therefore $v_k v^k = |v^k|^2$) decreases rapidly for large $|k|$.

(c) The imaginary part $\operatorname{Im} h(x) \neq 0$ if $\operatorname{Im} x \neq 0$, for all x in the domain of regularity. Thus, if $h(x)$ posseses any zeroes in that domain, they are real and $< \mu$.

(d) For real $x < \mu$, $h'(x) > 0$. So there is at most one zero x_0. The necessary and sufficient condition for this zero to exist is that $h(\mu) := h(\mu - 0) > 0$. Physically speaking, the V-particle is stable in this case; its mass (energy) being given by

$$M = m + x_0 \qquad (h(x_0) = 0)\,. \tag{A 1.1}$$

The renormalization constant Z is defined by

$$Z := h'(x_0)^{-1} > 0\,. \tag{A 1.2}$$

(Under our assumptions on the function v^k we have in addition $Z < 1$.)

(e)
$$h(x^*) = h(x)^*\,. \tag{A 1.3}$$

We define the (normalized) "vertical discontinuity" of any function $\psi(x)$ by

$$(\varrho\,\psi)(x) \equiv \varrho[\psi(x)] := -(2\pi i)^{-1}[\psi(x+i\,o) - \psi(x-i\,o)]\,, \tag{A 1.4}$$

provided the right-hand side exists at least as a distribution. For example, if ξ is a real parameter, we have

$$\varrho\left[\frac{1}{x-\xi}\right] = \delta(x-\xi)\,. \tag{A 1.5}$$

Applying this to the definition (14) of $h(x)$, it follows that

$$\varrho[h(x)] \equiv -\frac{1}{\pi}\operatorname{Im} h(x+i\,o) = -\int dk\, v_k v^k\, \delta(k_0 - x) \quad \text{for real } x\,. \tag{A 1.6}$$

(Obviously, $\varrho[h(x)]$ decreases rapidly for $x \to +\infty$ due to our assumptions on v^k.) Consequently,

$$\varrho[h(x)] = 0 \quad \text{for real } x < \mu\,. \tag{A 1.7}$$

Using these formulas, we may write

$$h(x) = x + m - M_0 + \int_\mu^\infty dx'\,(x-x')^{-1}\,\varrho[h(x')]\,, \tag{A 1.8}$$

which is the most convenient form of the PFD of $h(x)$.

From these properties of $h(x)$ one easily obtains the following PFD of the reciprocal $h(x)^{-1}$ by using Cauchy's and Liouville's theorems:

$$h(x)^{-1} = \Theta(h(\mu))\,Z\,(x-x_0)^{-1} + \int_\mu^\infty dx'\,(x-x')^{-1}\,\varrho[h(x')^{-1}]\,, \tag{A 1.9}$$

[14] H. v. DEWITZ, Diplomarbeit (in preparation).

where the pole term is present only if the V-particle is stable, which we have roughly indicated by the step function $\Theta(h(\mu))$. According (A 1.3) and (A 1.4),

$$\varrho[h(x')^{-1}] = -|h(x'+i o)|^{-2} \varrho[h(x')] \quad \text{for real} \quad x' > \mu . \tag{A 1.10}$$

From the above remarks it follows that the right-hand side decreases rapidly as $x' \to +\infty$. The total discontinuity of $h(x)^{-1}$ across the real axis is [see (A 1.5), (A 1.7), (A 1.10)]

$$\varrho_1(x) =: \varrho[h(x)^{-1}] = \Theta(h(\mu)) Z \delta(x-x_0) - |h(x+i o)|^{-2} \varrho[h(x)], \quad (x \text{ real}). \tag{A 1.11}$$

This is the quantity occurring in Eqs. (30), (31) of the text. It can be used to absorb the V-particle pole term of (A. 1.9) into the integral:

$$h(x)^{-1} = \int_{-\infty}^{+\infty} dx' (x-x')^{-1} \varrho_1(x') , \tag{A 1.12}$$

so that it is not necessary to distinguish explicitly between the cases of a stable and of an unstable V-particle.

The following formula will be used later:

$$h(y)^{-1} h(x-y)^{-1} = N(y,x) + N(x-y,x) . \tag{A 1.13}$$

It is easily proved by considering both sides as functions of the complex variable y and comparing the singularities and the asymptotic behaviour [13].

2. *Evaluation of some Matrix Elements in the V-sector*

In order to evaluate the matrix elements which occur in Eq. (59), we proceed as indicated at the end of Sect. 3. As an example, we calculate the matrix element

$$(V^* \omega, (y-H)^{-1} g(x-H) V^* \omega) , \tag{A 2.1}$$

where

$$g(x) := (x+m-M_0) h(x)^{-1} . \tag{A 2.2}$$

In the limit $y \to x - l_0$ it is identical with a matrix element contributing to the quantity d of Eq. (59). The PFD of $g(x)$ is obtained by noting that [see (A 1.12)]

$$1 = \lim_{x\to\infty} x\, h(x)^{-1} = \lim_{x\to\infty} \int_{-\infty}^{+\infty} dx'\, x(x-x')^{-1} \varrho_1(x') = \int_{-\infty}^{+\infty} dx'\, \varrho_1(x') , \tag{A 2.3}$$

hence

$$g(x) = \int_{-\infty}^{+\infty} dx'\, \varrho_1(x') (x-x'+x'+m-M_0) (x-x')^{-1} = 1 + \int_{-\infty}^{+\infty} dx'\, \varrho_1(x') (x'+m-M_0) (x-x')^{-1} . \tag{A 2.4}$$

From this it follows that

$$(y-H)^{-1} g(x-H) = (y-H)^{-1} + \int_{-\infty}^{+\infty} dx'\, \varrho_1(x') (x'+m-M_0) (y-H)^{-1} (x-H-x')^{-1} . \tag{A 2.5}$$

Using the PFD

$$(y-H)^{-1} (x-H-x')^{-1} = (x-y-x')^{-1} [(y-H)^{-1} - (x-H-x')^{-1}] , \tag{A 2.6}$$

we have from (A 2.4)

$$(y-H)^{-1} g(x-H) = (y-H)^{-1} g(x-y) - \int_{-\infty}^{+\infty} dx'\, \varrho_1(x') (x'+m-M_0) (x-y-x')^{-1} (x-H-x')^{-1} . \tag{A 2.7}$$

This is a PFD in the "variable" H. The matrix element (A 2.1) is then immediately obtained by means of Eq. (15):

$$(V^* \omega, (y-H)^{-1} g(x-H) V^* \omega) = h(y-m)^{-1} g(x-y) - N_2(x-y, x-m) \tag{A 2.8}$$

with

$$N_2(y,x) := \int_{-\infty}^{+\infty} dx'\, \varrho_1(x') (x'+m-M_0) (y-x')^{-1} h(x-x')^{-1} \tag{A 2.9 a}$$

$$= -D(x) + (y+m-M_0) N(y,x) . \tag{A 2.9 b}$$

This formula, together with (A 1.13) and (A 2.2), yields the final result

$$(V^* \omega, (y-H)^{-1} g(x-H) V^* \omega) = (x-y+m-M_0) N(y-m, x-m) + D(x-m). \tag{A 2.10}$$

In a similar manner, we get

$$(V^* \omega, h(x-H)^{-1} a^k N^* \omega) = v^k N(x-k_0-m, x-m) , \tag{A 2.11}$$

$$a := (a^l N^* \omega, h(x-H)^{-1} a^k N^* \omega) = \delta_l^k h(x-k_0-m)^{-1} + v^k v_l (k_0-l_0)^{-1} \{N(x-k_0-m, x-m) - N(x-l_0-m, x-m)\} , \tag{A 2.12}$$

$$c := v_l (V^* \omega, (x-H-l_0)^{-1} h(x-H)^{-1} a^k N^* \omega) = v^k v_l (x-k_0-l_0-m)^{-1} \{N(x-k_0-m, x-m) + N(x-l_0-m, x-m)\} , \tag{A 2.13}$$

$$(a^l N^* \omega, g(x-H) V^* \omega) = v_l N_2(x-l_0-m, x-m) . \tag{A 2.14}$$

Finally, we have to calculate the function $d(x-m)$ defined by (52). The matrix element of $g(x-H)$ occurring in the definition is easily evaluated by means of (A 2.4) and (15), the result being

$$d(x-m) = -1-2 \int_{-\infty}^{+\infty} dx' \, \varrho_1(x') \, (x'+m-M_0) \, h(x-m-x')^{-1} \tag{A 2.15 a}$$

$$= -1-2 \lim_{y\to\infty} y \, N_2(y, x-m) \tag{A 2.15 b}$$

[see (A 2.9 a)]. The right-hand side can be expressed by $D(x-m)$ as follows. From (A 2.9 b) one immediately derives the algebraic relation

$$(x+2m-2M_0) \, N(y,x) + N_2(x-y,x) - N_2(y,x) = (x-y+m-M_0) \, [N(x-y,x) + N(y,x)] = g(x-y) \, h(y)^{-1}. \tag{A 2.16}$$

In the last step, (A 1.13) and (A 2.2) have been used. Multiplying by y and taking the limit $y \to \infty$, we obtain, using (A 2.15 b):

$$(x+2m-2M_0) \, D(x) + \tfrac{1}{2} [1+d(x)] + \tfrac{1}{2} [1+d(x)] = 1 \tag{A 2.17}$$

or

$$d(x) = -(x+2m-2M_0) \, D(x) \, . \tag{A 2.18}$$

All quantities occurring in (59) have now been expressed in terms of the standard integrals N and D. By substitution into (59) it is easily seen that the algebraically calculated quantity r_l^k equals that given in Eq. (32) which was derived by solving the integral Equation (25).

Level-Crossing-Untersuchung des 6p $^2P_{3/2}$-Terms im Au I-Spektrum durch Resonanzstreuung von Licht im elektrischen und magnetischen Feld zur Bestimmung der Hyperfeinstrukturtermordnung

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Stark Effect Investigation of the 6p $^2P_{3/2}$*-State in* Au I*-Spectrum in Order to Determine the Signs of the Hyperfine Structure Constants*

Level crossing technique, applying parallel electric and magnetic fields, has been used to determine the signs of the hyperfine structure constants A and B as well as the magnitude of the Stark-constant β. From the Stark-shift of the signal at 45 Oe in the wing of the Hanle-effect and of the level-crossing near 140 Oe to higher magnetic field strengths the ratios $A/\beta > 0$ and $B/\beta > 0$ could be determined. An estimation of β by calculating the electric dipole matrix elements with Coulomb approximation yields a positive sign. Both signals and their shift with electric field then can be explained with the following results:

$A = +14.0(5)$ Mc/sec; $B = +327.6(1.6)$ Mc/sec and $\beta = +5.7(3)$ (kc/sec) (kV/cm)$^{-2}$.

I. Einleitung

In einem Resonanzstreuexperiment im äußeren Magnetfeld konnten die Beträge des magnetischen Aufspaltungsfaktors A, der Kernquadrupolkopplungskonstanten B und das Vorzeichen von B/A bestimmt werden [1]. Durch ein zusätzliches elektrisches Feld kann die Hyperfeinstruktur-Termordnung gemessen werden, wenn der Einfluß des elektrischen Feldes [2] auf die Zeemanunterniveaus bekannt ist. Im Zusammenhang mit der Frage nach Termstörungen beziehungsweise Polarisation der Elektronenhülle durch das Leuchtelektron [3] oder durch den Kern [4] besteht Interesse an der Bestimmung der Vorzeichen von A und B, zumal Hyperfeinstrukturuntersuchungen an weiteren Konfigurationen [5] zum Vergleich vorliegen.

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